AIAA JOURNAL
Vol. 39, No. 11, November 2001

# Exact and Accurate Solutions in the Approximate Reanalysis of Structures

Uri Kirsch* and Panos Y. Papalambros†
*University of Michigan, Ann Arbor, Michigan 48109*

**Combined approximations (CA) is an efficient method for reanalysis of structures where binomial series terms are used as basis vectors in reduced basis approximations. In previous studies high-quality approximations have been achieved for large changes in the design, but the reasons for the high accuracy were not fully understood. In this work some typical cases, where exact and accurate solutions are achieved by the method, are presented and discussed. Exact solutions are obtained when a basis vector is a linear combination of the previous vectors. Such solutions are obtained also for low-rank modifications to structures or scaling of the initial stiffness matrix. In general the CA method provides approximate solutions, but the results presented explain the high accuracy achieved with only a small number of basis vectors. Accurate solutions are achieved in many cases where the basis vectors come close to being linearly dependent. Such solutions are achieved also for changes in a small number of elements or when the angle between the two vectors representing the initial design and modified design is small. Numerical examples of various changes in cross sections of elements and in the layout of the structure show that accurate results are achieved even in cases where the series of basis vectors diverges.**

## I. Introduction

MULTIPLE repeated analyses are needed in various design and optimization problems. In general, the structural response cannot be expressed explicitly in terms of the structure properties, and structural analysis involves the solution of a set of simultaneous equations. Reanalysis methods are intended to efficiently analyze structures modified due to changes in the design.

Approximate reanalysis methods have been used extensively in structural optimization to reduce the number of exact analyses and the overall computational cost during the solution process. The combined approximations (CA) method developed recently is considered in this paper. The method combines several concepts and methods such as reduced basis, series approximations, matrix factorization and Gram–Schmidt orthonormalization. These and other methods are used to achieve effective solution procedures. The effectiveness of the method in various optimization problems has been demonstrated in previous studies.[1–5] Initially the CA method was used only for linear reanalysis models. Recently, the method has been used successfully also in eigenvalue[6] and nonlinear analysis[7] problems. Applications of the method in a large variety of problems are discussed elsewhere.[8–11]

High-quality approximations of the structural response for large changes in the design have been achieved in previous studies, but the reasons for the high accuracy were not fully understood. In this paper some typical cases, where exact and accurate solutions are achieved by the CA method, are presented and discussed. In general the CA method provides exact solutions, but the results presented in the paper explain the high accuracy achieved with only a small number of basis vectors. The solution procedure is briefly described in Sec. II. Three typical cases, where exact solutions are achieved by the CA method, are introduced and discussed in Sec. III. Exact solutions are obtained when a basis vector is a linear combination of the previous vectors. Such solutions are obtained also for low-rank modifications to structures or scaling of the initial stiffness matrix. Various cases of accurate solutions are discussed in Sec. IV. Convergence properties of the series of basis vectors and the series of the CA terms are presented, and criteria intended to evaluate the errors in the approximations are introduced. Accurate solutions are achieved in many cases where the basis vectors come close to being linearly dependent. Such solutions are achieved also for changes in a small number of elements or when the angle between the two vectors representing the initial design and modified design is small. Numerical examples illustrating the accuracy of the results are included in Sec. V.

Received 27 April 2000; accepted for publication 5 April 2001. 

*Visiting Professor, Department of Mechanical Engineering and Applied Mechanics; on leave from Department of Civil Engineering, Technion—Israel Institute of Technology, 32000 Haifa, Israel.

†Professor, Department of Mechanical Engineering and Applied Mechanics.

## II. Approximate Reanalysis by the CA Method

Consider an initial design with stiffness matrix $\boldsymbol{K}^*$ and corresponding displacements $\boldsymbol{r}^*$ computed by the stiffness analysis equations:

$$\boldsymbol{K}^*\boldsymbol{r}^* = \boldsymbol{R} \tag{1}$$

For simplicity of presentation the load vector $\boldsymbol{R}$ is assumed to be constant, but the procedure presented is suitable also when the elements of $\boldsymbol{R}$ are functions of the design variables. Assume a change in the design so that the modified stiffness matrix is given by

$$\boldsymbol{K} = \boldsymbol{K}^* + \Delta\boldsymbol{K} \tag{2}$$

where $\Delta\boldsymbol{K}$ is the corresponding change in the stiffness matrix. The object is to evaluate the modified displacements $\boldsymbol{r}$ due to various changes $\Delta\boldsymbol{K}$ efficiently and accurately, without solving the complete set of modified implicit equations:

$$\boldsymbol{K}\boldsymbol{r} = (\boldsymbol{K}^* + \Delta\boldsymbol{K})\boldsymbol{r} = \boldsymbol{R} \tag{3}$$

Evaluation of the modified displacements by the CA method is briefly described in the following.

We assume that the displacement vector of a modified design can be approximated by a linear combination of $s$ linearly independent basis vectors as

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{4}$$

where $\boldsymbol{r}_B$ is the matrix of $s$ basis vectors and $\boldsymbol{y}$ is a vector of the $s$ coefficients to be determined. Defining matrix $\boldsymbol{B}$ by

$$\boldsymbol{B} = \boldsymbol{K}^{*-1}\Delta\boldsymbol{K} \tag{5}$$

the basis vectors are then given by the terms of the binomial series

$$\boldsymbol{r}_1 = \boldsymbol{r}^*, \qquad \boldsymbol{r}_2 = -\boldsymbol{B}\boldsymbol{r}^*, \qquad \boldsymbol{r}_3 = \boldsymbol{B}^2\boldsymbol{r}^* \cdots \boldsymbol{r}_s = -\boldsymbol{B}^{s-1}\boldsymbol{r}^* \tag{6}$$

Calculation of the series terms involves only forward and backward substitutions in cases where the initial stiffness matrix $\boldsymbol{K}^*$ is given in a decomposed form from the initial analysis.[5]

To determine the vector of coefficients $\boldsymbol{y}$, the approximate displacements of Eq. (4) are substituted into the modified analysis equations (3). Premultiplying the resulting equation by $\boldsymbol{r}_B^T$ yields

$$\boldsymbol{K}_R\boldsymbol{y} = \boldsymbol{R}_R \tag{7}$$

where

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{8}$$

For cases where $s$ is much smaller than the number of degrees of freedom $n$, the approximate displacement vector can be evaluated by solving the smaller $(s \times s)$ system in Eq. (7) for $\boldsymbol{y}$ instead of computing the exact solution by solving the large $(n \times n)$ system in Eq. (3). The final displacements are then computed for the given $\boldsymbol{y}$ by Eq. (4).

In summary, evaluation of the modified displacements by the CA method involves the following steps:

1) The modified stiffness matrix $\boldsymbol{K}$ is first introduced [Eq. (3)]. Because $\boldsymbol{K}^*$ is already given, this step involves only calculation of $\Delta\boldsymbol{K}$.

2) The basis vectors $\boldsymbol{r}_i$ are calculated by Eq. (6). Calculation of each basis vector involves only forward and backward substitutions.

3) The reduced matrix $\boldsymbol{K}_R$ and the reduced vector $\boldsymbol{R}_R$ are calculated by Eq. (8).

4) The vector of unknown coefficients $\boldsymbol{y}$ is calculated by solving the set of $(s \times s)$ equations (7).

5) The modified displacements $\boldsymbol{r}$ are evaluated by Eq. (4).

The solution process is based on results of a single exact analysis, and it is suitable for different types of structures and design changes. The method is easy to implement; it can be used readily with a general finite element program; and calculation of derivatives is not required.

The efficiency of reanalysis by the CA method, compared with complete analysis of the modified design, can be measured by various criteria, e.g., the CPU effort or the number of algebraic operations. It is then possible to relate the computational effort to various parameters such as the number of degrees of freedom, the number of basis vectors considered, and the accuracy of the results. It was found that calculation of each basis vector involves about 2% of the CPU time needed for complete analysis. In many cases a small number of basis vectors is sufficient to achieve adequate accuracy. For moderate changes in the design, two to three vectors are often sufficient, whereas five to six vectors might be needed for large changes. Considering the latter number of basis vectors, results for various problems showed that the total CPU effort, compared with complete analysis of the modified design, has been reduced by more than 75%.

## III. Exact Solutions

In this section the following three typical cases, where exact solutions are obtained by the CA method, are presented:

1) A general case, where a basis vector is a linear combination of the previous vectors, is developed in Sec. III.A. In many cases, where the basis vectors come close to being linearly dependent, accurate solutions are achieved by the CA method.

2) The case of low-rank modifications to structures, where the number of modified elements in the stiffness matrix is small, is presented in Sec. III.B. The exact solution is achieved by the CA method if one basis vector is introduced for each changed member. If some of the basis vectors are linearly dependent, the exact solution is achieved with a smaller number of vectors. This procedure is efficient when the number of changed members is much smaller than the number of degrees of freedom.

3) The common case of scaling of the initial stiffness matrix, where exact solutions are obtained by consideration of a single basis vector, is presented in Sec. III.C. In many cases, where the angle between the two vectors representing the initial design and modified design is small, accurate solutions are achieved by the CA method.

### A. Linearly Dependent Basis Vectors

A general case of changes in the structure, where the exact solution is obtained by the CA method, is presented in this section. To obtain a convenient expression for the exact solution of the modified design, premultiply Eq. (3) by $\boldsymbol{K}^{*-1}$ and substitute Eqs. (1) and (5) to obtain

$$(\boldsymbol{I} + \boldsymbol{B})\boldsymbol{r} = \boldsymbol{r}^* \tag{9}$$

Premultiplying Eq. (9) by $(\boldsymbol{I} + \boldsymbol{B})^{-1}$ gives the exact modified displacements

$$\boldsymbol{r} = (\boldsymbol{I} + \boldsymbol{B})^{-1}\boldsymbol{r}^* \tag{10}$$

To achieve a convenient expression for the approximate displacements in terms of the assumed $s$ basis vectors, substitute the expressions of the basis vectors [Eq. (6)] into Eq. (4):

$$\boldsymbol{r} = y_1\boldsymbol{r}^* - y_2\boldsymbol{B}\boldsymbol{r}^* + y_3\boldsymbol{B}^2\boldsymbol{r}^* - \cdots + y_s\boldsymbol{B}^{s-1}\boldsymbol{r}^* \tag{11}$$

Assuming that the approximate solution of Eq. (11) involving $s$ terms is equal to the exact solution of Eq. (10), premultiplying both equations by $(\boldsymbol{I} + \boldsymbol{B})$ and rearranging, we obtain the linear expression for the additional term:

$$\boldsymbol{r}_{s+1} = \sum_{i=1}^{s} a_i\boldsymbol{r}_i \tag{12}$$

where $a_i$ are scalar multipliers given by

$$a_1 = (y_1 - 1)/y_s, \qquad a_i = (y_i - y_{i-1})/y_s, \qquad i = 2, 3, \ldots, s \tag{13}$$

Equation (12) shows that, when the reduced basis expression with $s$ terms [Eq. (11)] is equal to the exact solution, then the $s+1$ basis vector is a linear combination of the previous $s$ vectors. That is, the $s+1$ basis vectors are linearly dependent.

### B. Simultaneous Rank-One Changes

Exact methods are efficient in cases of low-rank modifications to structures and are applicable to situations where the number of modified elements in the stiffness matrix is limited. These methods are usually based on the Sherman–Morrison[12] and Woodbury[13] formulas for the update of the inverse of a matrix. It has been shown recently[14] that various reanalysis methods may be viewed as variants of these formulas.

Consider, for example, the typical case of simultaneous changes in $m$ truss members. The exact solution is obtained if one basis vector is selected for each changed member[3]:

$$\boldsymbol{r}_i = \boldsymbol{K}^{*-1}\Delta\boldsymbol{K}_i\boldsymbol{r}^*, \qquad i = 1, \ldots, m \tag{14}$$

where $\Delta\boldsymbol{K}_i$ is the contribution of the $i$th member to $\Delta\boldsymbol{K}$. The exact solution is given by

$$\boldsymbol{r} = \boldsymbol{r}^* + \sum_{i=1}^{m} y_i\boldsymbol{r}_i \tag{15}$$

where $\boldsymbol{r}^*$ is the vector of initial displacements. This procedure is efficient when the number of changed members is much smaller than the number of degrees of freedom. Exact solutions achieved by the CA method and the Sherman–Morrison–Woodbury formulas in such cases are equivalent. If some of the basis vectors are linearly dependent, the exact solution is obtained with a smaller number of vectors.

### C. Scaling of the Initial Stiffness Matrix

Scaling of the initial stiffness matrix $\boldsymbol{K}^*$ is carried out by multiplying the latter matrix by a positive scaling multiplier $\mu$ to obtain the modified matrix:

$$\boldsymbol{K} = \mu\boldsymbol{K}^* \tag{16}$$

From Eqs. (1), (3), and (16) it is clear that the exact displacements after scaling are given by

$$\boldsymbol{r} = \mu^{-1}\boldsymbol{r}^* \tag{17}$$

The condition of Eq. (16) requires linear dependence of the stiffness matrix on the change in the design. In general, the elements of $\boldsymbol{K}$ are some nonlinear functions of the design variables. A typical case where the condition of Eq. (16) is satisfied is scaling of the cross sections or the geometry of a truss structure, where the lengths of all elements are multiplied by $\mu$ and their direction is unchanged.

Consider the case where the modified design is a scaled design $\mu\boldsymbol{K}^*$, as given by Eq. (16). Then, from Eq. (5),

$$\boldsymbol{B} = \boldsymbol{K}^{*-1}\Delta\boldsymbol{K} = (\mu - 1)\boldsymbol{I} \tag{18}$$

where $\boldsymbol{I}$ is the identity matrix. The resulting basis vectors [Eqs. (6)] become linearly dependent:

$$\boldsymbol{r}_1 = \boldsymbol{r}^*, \qquad \boldsymbol{r}_2 = -(\mu - 1)\boldsymbol{r}^*, \qquad \boldsymbol{r}_3 = (\mu - 1)^2\boldsymbol{r}^* \cdots \tag{19}$$

Thus, the exact modified displacements are determined directly by the first basis vector, and no approximations are needed. Consideration of the single basis vector with a coefficient $y_1 = \mu^{-1}$ will provide the exact solution as given by Eq. (17).

## IV. Accuracy Considerations

### A. Error Evaluation

The various cases of exact solutions discussed in Sec. III explain the high accuracy achieved by the CA method. In this section convergence considerations related to the series of basis vectors and the series of the CA terms are presented, and criteria intended to evaluate the errors in the approximations are introduced.

The series of basis vectors [Eqs. (6)] converges if and only if

$$\lim_{k \to \infty} \boldsymbol{B}^k = \boldsymbol{0} \tag{20}$$

A sufficient criterion for the convergence of the series is that $\|\boldsymbol{B}\| < 1$, where $\|\boldsymbol{B}\|$ is the norm of $\boldsymbol{B}$. It can be shown that $\rho(\boldsymbol{B}) \le \|\boldsymbol{B}\|$, where $\rho(\boldsymbol{B})$ is the spectral radius, i.e., the largest eigenvalue of matrix $\boldsymbol{B}$. From the preceding, a sufficient condition for convergence is $\rho(\boldsymbol{B}) < 1$.

It is convenient to express the change in the design $\Delta\boldsymbol{K}$ [Eq. (2)] as

$$\Delta\boldsymbol{K} = \alpha\Delta\boldsymbol{K}^* \tag{21}$$

where $\Delta\boldsymbol{K}^*$ is a matrix representing the direction of change and $\alpha$ is a scalar multiplier describing the magnitude of change in the design. In the solution process, the basis vectors $\boldsymbol{r}_i$ are determined by Eq. (6) and are multiplied by the corresponding scalars $y_i$ to obtain the final terms $y_i\boldsymbol{r}_i$ [Eq. (4)]. Multiplying a basis vector $\boldsymbol{r}_i$ by any scalar will not change the approximate solution (but only the scalar $y_i$). Therefore, identical basis vectors can be selected for any given $\Delta\boldsymbol{K}^*$ and different $\alpha$ values. In cases where the elements of the basis vectors become very large due to large $\Delta\boldsymbol{K}$ values, it is possible to normalize a basis vector $\boldsymbol{r}_i$ by dividing it by an arbitrary reference element of the vector (e.g., the first element $r_{1i}$) to obtain the normalized vector:

$$\boldsymbol{r}_{Ni} = \boldsymbol{r}_i / r_{1i} \tag{22}$$

This operation scales the first element of the vector to unity and, as noted earlier, does not change the approximate solution.

To study the accuracy of the approximations, an uncoupled set of new basis vectors $\boldsymbol{V}_i$ $(i = 1, \ldots, s)$ is introduced using a Gram–Schmidt orthogonalization and normalization method.[5,7] The new vectors are determined by the original ones $\boldsymbol{r}_i$ from

$$\boldsymbol{V}_1 = \left|\boldsymbol{r}_1^T\boldsymbol{K}\boldsymbol{r}_1\right|^{-\frac{1}{2}}\boldsymbol{r}_1 \tag{23}$$

$$\bar{\boldsymbol{V}}_i = \boldsymbol{r}_i - \sum_{j=1}^{i-1}\left(\boldsymbol{r}_i^T\boldsymbol{K}\boldsymbol{V}_j\right)\boldsymbol{V}_j, \qquad \boldsymbol{V}_i = \left|\bar{\boldsymbol{V}}_i^T\boldsymbol{K}\bar{\boldsymbol{V}}_i\right|^{-\frac{1}{2}}\bar{\boldsymbol{V}}_i$$

$$i = 2, \ldots, s \tag{24}$$

where $\bar{\boldsymbol{V}}_i$ and $\boldsymbol{V}_i$ are the $i$th nonnormalized and normalized vectors, respectively. Defining the matrix $\boldsymbol{V}_B$ of new basis vectors and the vector $\boldsymbol{z}$ of new coefficients, the reduced system of Eq. (7) becomes uncoupled and the final displacements are given by the explicit expression

$$\boldsymbol{r} = \boldsymbol{V}_B\boldsymbol{z} = \boldsymbol{V}_B\left(\boldsymbol{V}_B^T\boldsymbol{R}\right) \tag{25}$$

Equation (25) can be expressed as an additively separable quadratic function of the basis vectors $\boldsymbol{V}_i$ by

$$\boldsymbol{r} = \sum_{i=1}^{s}\boldsymbol{V}_i\left(\boldsymbol{V}_i^T\boldsymbol{R}\right) \tag{26}$$

One advantage in using the new vectors is that all expressions for evaluating the displacements are explicit functions of the original basis vectors. Calculation of any new basis vector $\boldsymbol{V}_i$ leads to an additional term in the displacements expression [Eq. (26)] that is a function of the original vectors $\boldsymbol{r}_j$ $(j = 1, 2, \ldots, i)$. As a result, additional vectors can be considered without modifying the calculations that were carried out already.

For any assumed number of basis vectors, the results obtained by considering either the original set of basis vectors or the new set of uncoupled basis vectors are identical. Whereas the normalized vectors $\boldsymbol{V}_i$ are of similar magnitude, the values of the $z_i$ coefficients and, therefore, the corresponding terms of the series of Eq. (26) are gradually decreased. It will be shown by numerical examples in Sec. V that transforming the binomial series terms [Eq. (6)] into the terms of the CA series [Eq. (26)] provides accurate solutions even in cases where the binomial series diverges.

The errors in the results for a specific number $s$ of basis vectors can be evaluated by assessing the size of the elements of the $s$th term of the approximate displacements:

$$\boldsymbol{r}^{(s)} = \boldsymbol{V}_s\left(\boldsymbol{V}_s^T\boldsymbol{R}\right) \tag{27}$$

If the solution process converges, the size of the elements of the vector $\boldsymbol{r}^{(s)}$ in Eq. (27) can be used as a convergence criterion, namely

$$E_r \equiv \frac{\left\|\boldsymbol{r}^{(s)}\right\|}{\left\|\sum_{i=1}^{s}\boldsymbol{r}^{(i)}\right\|} = \frac{\left\|\boldsymbol{V}_s\left(\boldsymbol{V}_s^T\boldsymbol{R}\right)\right\|}{\left\|\sum_{i=1}^{s}\boldsymbol{V}_i\left(\boldsymbol{V}_i^T\boldsymbol{R}\right)\right\|} \le E_r^U \tag{28}$$

where $E_r^U$ is a small number and $\|\,\|$ is the Euclidean norm. Because the normalized vectors $\boldsymbol{V}_i$ are of similar magnitude, whereas the values of the $z_i$ coefficients are gradually decreased, an alternative convergence criterion is

$$E_z \equiv \frac{|z_s|}{\left|\sum_{i=1}^{s} z_i\right|} = \frac{\left|\boldsymbol{V}_s^T\boldsymbol{R}\right|}{\left|\sum_{i=1}^{s}\boldsymbol{V}_i^T\boldsymbol{R}\right|} \le E_z^U \tag{29}$$

where $E_z^U$ is again a small number.

### B. Accurate Solutions

It was noted previously that in general the CA method provides approximate solutions but that accurate results are often achieved with only a small number of basis vectors. This section presents some typical cases where such results are expected.

*Nearly Dependent Basis Vectors*

It was shown in Sec. III.A that exact solutions are obtained in cases where the basis vectors are linearly dependent. As a result, it is expected that accurate solutions will be achieved in cases where the basis vectors come close to being linearly dependent. Two basis vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}_{i+1}$ are close to being linearly dependent if

$$\cos\beta_{ii+1} = \frac{\boldsymbol{r}_i^T\boldsymbol{B}^*\boldsymbol{r}_i}{|\boldsymbol{r}_i|\left|\boldsymbol{B}^*\boldsymbol{r}_i\right|} \simeq 1 \tag{30}$$

where $\beta_{ii+1}$ is the angle between the two vectors. It can be noted [Eq. (21)] that, for any given $\Delta\boldsymbol{K}^*$, the angle $\beta_{ii+1}$ is independent of the scalar $\alpha$. It is shown by numerical examples that the basis vectors determined by the CA method satisfy the condition of Eq. (30), as the basis vectors index $i$ is increased, even for very large changes in the design.

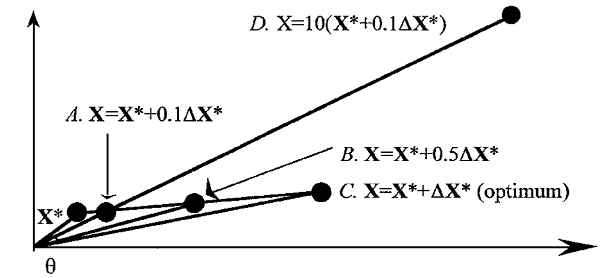


**Fig. 1 Various modified designs shown in the space of $X^*$ and $\Delta X^*$.**

*Changes in Limited Number of Elements*

It was noted in Sec. III.B that, for simultaneous $m$ rank-one changes, exact solutions are obtained by the CA method if one basis vector is selected for each changed member. If some of the basis vectors are linearly dependent, the exact solution is achieved for a smaller number of vectors. It has been observed that, in many cases where small numbers of elements are changed, nearly exact solutions are achieved by the CA method with a small number of basis vectors. Results will be demonstrated by numerical examples in Sec. V.

*Nearly Scaled Designs*

Consider the case where the change in the stiffness matrix $\Delta \boldsymbol{K}$ [Eq. (21)] can be expressed in terms of corresponding change in the design variables $\boldsymbol{X}$ by

$$\boldsymbol{X} = \boldsymbol{X}^* + \Delta \boldsymbol{X} = \boldsymbol{X}^* + \alpha \Delta \boldsymbol{X}^* \tag{31}$$

Both the direction of change $\Delta \boldsymbol{X}^*$ and the magnitude of change $\alpha$ may significantly affect the accuracy of the approximations. This effect can be quantified by the cosine of the angle $\theta$ between the vector of the modified design $\Delta \boldsymbol{X}$ and the vector of initial design $\Delta \boldsymbol{X}^*$:

$$\cos\theta = (\boldsymbol{X}^T \boldsymbol{X}^*)/(|\boldsymbol{X}|\,|\boldsymbol{X}^*|) \tag{32}$$

where $|\boldsymbol{X}|$ denotes the absolute value of $\boldsymbol{X}$. Figure 1 shows that various designs, obtained by scaling a certain modified design, provide identical $\theta$ angles. For example, the two modified designs A at $\boldsymbol{X} = \boldsymbol{X}^* + 0.1\Delta \boldsymbol{X}^*$ and D at $\boldsymbol{X} = 10(\boldsymbol{X}^* + 0.1\Delta \boldsymbol{X}^*)$ correspond to an identical $\theta$. It will be shown in Sec. V that high accuracy is achieved with a small number of basis vectors for designs A (representing a small change in the design) and D (representing a very large change in the design) because both correspond to a small $\theta$ value. More basis vectors are needed for designs B and C that correspond to larger $\theta$. In the present discussion the space formed by the vectors $\boldsymbol{X}^*$ and $\boldsymbol{X}$ is considered. It should be emphasized that, for the complete design space, smaller $\theta$ values do not always guarantee better approximations.

For any given direction vector $\Delta \boldsymbol{X}^*$, the magnitude of change $\alpha$ determines the value of $\theta$ and the accuracy of the results. The larger $\alpha$ is, the larger is the angle $\theta$, and more basis vectors might be required to achieve adequate accuracy. For $\theta = 0$, an exact solution is achieved by scaling the initial displacements $\boldsymbol{r}^*$. Because accurate results are expected for small angles $\theta$, it might prove useful to apply the angle constraints[15]

$$\theta^L \le \theta \le \theta^U \tag{33}$$

where $\theta^L$ and $\theta^U$ are predetermined limits.

Common limitations on changes in the design are the move limit constraints

$$\Delta \boldsymbol{X}^L \le \Delta \boldsymbol{X} \le \Delta \boldsymbol{X}^U \tag{34}$$

where $\Delta \boldsymbol{X}^L$ and $\Delta \boldsymbol{X}^U$ are predetermined lower and upper limits, respectively, on the design variable changes. An alternative approach, used in trust region algorithms,[16] is to restrict the solutions to some region around $\boldsymbol{X}^*$ by constraints of the form

$$\|\Delta \boldsymbol{X}\| \le \Delta \tag{35}$$

where $\Delta$ is the radius of the region to be restricted. Constraints of the type of Eqs. (34) and (35) are generally effective only for local approximations (such as the Taylor series), where small changes in the design variables are assumed. They are not effective for the CA method applied to find accurate solutions for large changes in the design.

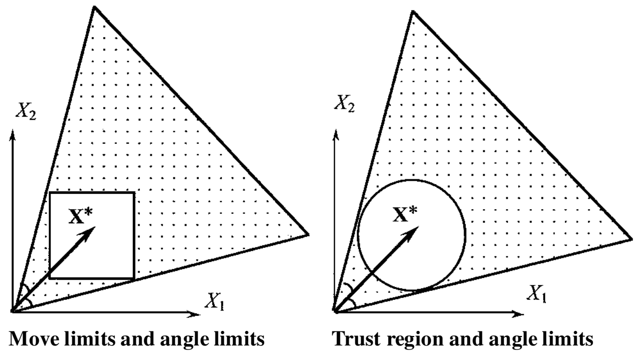


**Fig. 2 Two-dimensional presentation of limitations on design changes.**

Figure 2 shows that the constraints of Eq. (33) define a large conical region rather than the relatively small region defined by the local approximation constraints in the neighborhood of $\boldsymbol{X}^*$.

## V. Numerical Examples

### A. Convergence for Various Design Changes

Consider the classic 10-bar truss problem shown in Fig. 3, subjected to a single loading condition of two concentrated loads. The design variables are the member cross-sectional areas, the initial cross-sectional areas are all unity, the modulus of elasticity is 30,000, and the eight analysis unknowns are the horizontal and vertical displacements at joints 1, 2, 3, and 4, respectively. The stress constraints for all 10 members are $-25.0 \le \sigma \le 25.0$, and the minimum size constraints are $0.001 \le \boldsymbol{X}$. Assuming the weight as an objective function, the optimal design is

$$\boldsymbol{X}_{\text{opt}}^T = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

The line from the initial design to the optimal design is given by

$$\boldsymbol{X} = \boldsymbol{X}^* + \alpha \Delta \boldsymbol{X}^*$$

where $\Delta \boldsymbol{X}^*$ is defined as

$$\Delta \boldsymbol{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

For $\alpha = 1.0$ (the optimum) the change in the design is very large: members 1 and 3 are increased by 700%; member 4 is increased by 300%; members 7–9 are increased by 467%; and the topology is changed by effectively eliminating members 2, 5, 6, and 10 and, therefore, joint 2 (displacements 3 and 4). To illustrate the effect of various design changes on the accuracy of the results, four typical cases were considered (see Fig. 1 and Table 1):

1) Small change in the design (up to $-10$ and $+70\%$) and small angle $\theta$ ($\alpha = 0.1$, $\theta = 14$ deg). The modified design is given by $\boldsymbol{X} = \boldsymbol{X}^* + 0.1\Delta \boldsymbol{X}^*$ (design A).

2) Medium change in the design (up to $-50$ and $+350\%$) and medium angle $\theta$ ($\alpha = 0.5$, $\theta = 34$ deg). The modified design is given by $\boldsymbol{X} = \boldsymbol{X}^* + 0.5\Delta \boldsymbol{X}^*$ (design B).

3) Large change in the design (up to $-100$ and $+700\%$) and large angle $\theta$ (the optimum, $\alpha = 1.0$, $\theta = 41$ deg). The modified design is given by $\boldsymbol{X} = \boldsymbol{X}^* + \Delta \boldsymbol{X}^*$ (design C).

4) Very large change in the design (up to $+1600\%$ and small angle $\theta$ ($\theta = 14$ deg). The modified design is given by scaling design A by a factor of 10, $\boldsymbol{X} = 10(\boldsymbol{X}^* + 0.1\Delta \boldsymbol{X}^*)$ (design D).

The exact displacements for the initial design and the four cases of modified designs are given in Table 2. Note that the modified displacements are significantly different from the initial ones. Results obtained for the different modified designs and various numbers of basis vectors by the CA method are summarized in Table 3. An accurate solution (maximum displacement error of 0.05) is achieved

**Table 1 Modified cross sections for various modified designs (10-bar truss)**

| | Change for design ($\theta$, deg) | | | |
|---|---|---|---|---|
| Member | A (14) | B (34) | C (41) | D (14) |
| 1 | 1.700 | 4.500 | 8.000 | 17.000 |
| 2 | 0.900 | 0.500 | 0.001 | 9.000 |
| 3 | 1.700 | 4.500 | 8.000 | 17.000 |
| 4 | 1.300 | 2.500 | 4.000 | 13.000 |
| 5 | 0.900 | 0.500 | 0.001 | 9.000 |
| 6 | 0.900 | 0.500 | 0.001 | 9.000 |
| 7 | 1.467 | 3.333 | 5.667 | 14.667 |
| 8 | 1.467 | 3.333 | 5.667 | 14.667 |
| 9 | 1.467 | 3.333 | 5.667 | 14.667 |
| 10 | 0.900 | 0.500 | 0.001 | 9.000 |

**Table 2 Initial and modified exact displacements (10-bar truss)**

| | Displacement for design | | | |
|---|---|---|---|---|
| Initial | A | B | C | D |
| 2.34 | 1.37 | 0.52 | 0.30 | 0.14 |
| 5.58 | 3.56 | 1.49 | 0.90 | 0.36 |
| 2.83 | 1.77 | 0.77 | 0.52 | 0.18 |
| 12.65 | 8.25 | 3.65 | 2.17 | 0.83 |
| −3.17 | −2.10 | −0.98 | −0.60 | −0.21 |
| 13.13 | 8.65 | 3.90 | 2.40 | 0.87 |
| −2.46 | −1.45 | −0.55 | −0.30 | −0.15 |
| 6.01 | 3.89 | 1.62 | 0.90 | 0.39 |

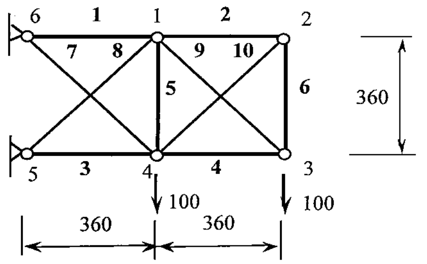


**Fig. 3 Ten-bar truss.**

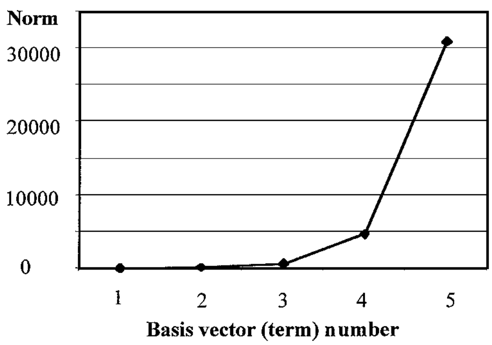


**Fig. 4 Norm of the original basis vectors $r_i$ (binomial series terms) for $\alpha = 1.0$.**

with only two basis vectors for designs A and D. Similar accuracy is achieved in both cases with identical small $\theta$ values, although the change in design D is much larger. The accuracy in design D is higher than that of design C, although the change in the latter design is smaller (but the angle $\theta$ is larger). For a given direction $\Delta X^*$, the number of basis vectors needed to achieve a certain accuracy is increased with $\alpha$. Accurate solution (again, maximum displacement error of 0.05) is achieved with three basis vectors for design B ($\alpha = 0.5$) and with four basis vectors for design C ($\alpha = 1$).

Consider again the results for design C ($\alpha = 1$). Figure 4 shows how the norm of the terms $\boldsymbol{r}_i$ of the series of basis vectors (the binomial series) increases as the basis vectors index is increased, and the series diverges. Figure 5 shows the norm of the uncoupled basis vectors $\boldsymbol{V}_i$, and Fig. 6 shows the coefficients $z_i$. Note that, whereas

**Table 3 Displacements for various modified designs (10-bar truss)**

| | Displacement by method | | | |
|---|---|---|---|---|
| | CA | | | |
| Design | 2[a] | 3 | 4 | Exact |
| A: $\theta = 14$ deg | 1.36 | | | 1.37 |
| (small change) | 3.59 | | | 3.56 |
| | 1.76 | | | 1.77 |
| | 8.23 | | | 8.25 |
| | −2.06 | | | −2.10 |
| | 8.62 | | | 8.65 |
| | −1.44 | | | −1.45 |
| | 3.92 | | | 3.89 |
| B: $\theta = 34$ deg | 0.50 | 0.52 | | 0.52 |
| (medium change) | 1.53 | 1.46 | | 1.49 |
| | 0.71 | 0.76 | | 0.77 |
| | 3.56 | 3.63 | | 3.65 |
| | −0.89 | −0.98 | | −0.98 |
| | 3.77 | 3.87 | | 3.90 |
| | −0.54 | −0.55 | | −0.55 |
| | 1.71 | 1.64 | | 1.62 |
| C: $\theta = 41$ deg | 0.28 | 0.29 | 0.29 | 0.30 |
| (large change) | 0.90 | 0.84 | 0.88 | 0.90 |
| | 0.41 | 0.45 | 0.47 | 0.52[b] |
| | 2.10 | 2.17 | 2.19 | 2.17[b] |
| | −0.53 | −0.61 | −0.62 | −0.60 |
| | 2.24 | 2.34 | 2.37 | 2.40 |
| | −0.30 | −0.31 | −0.31 | −0.30 |
| | 1.01 | 0.95 | 0.93 | 0.90 |
| D: $\theta = 14$ deg | 0.14 | | | 0.14 |
| (very large change) | 0.36 | | | 0.36 |
| | 0.18 | | | 0.18 |
| | 0.82 | | | 0.83 |
| | −0.21 | | | −0.21 |
| | 0.86 | | | 0.87 |
| | −0.14 | | | −0.15 |
| | 0.39 | | | 0.39 |

[a]Number of basis vectors. [b]Joint 2 is effectively eliminated.

**Table 4 Values of cos $\beta_{i,i+1}$: various modified designs and basis vectors (10-bar truss)**

| Design | $\beta_{1,2}$ | $\beta_{2,3}$ | $\beta_{3,4}$ | $\beta_{4,5}$ |
|---|---|---|---|---|
| A, B, C | 0.9989 | 0.9992 | 0.9994 | 0.9997 |
| D | 0.9998 | 0.9999 | 0.9999 | 0.9999 |

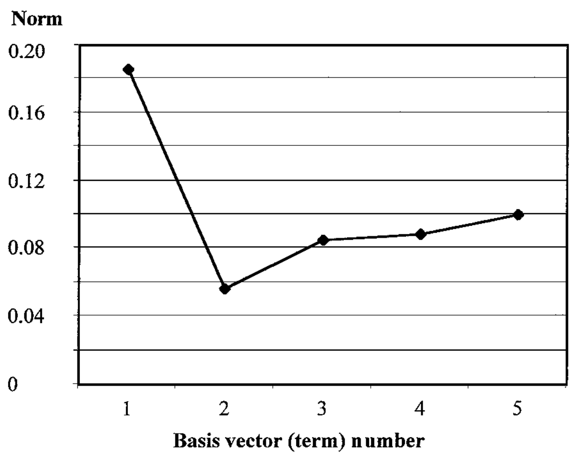


**Fig. 5 Norm of the uncoupled normalized basis vectors $V_i$ for $\alpha = 1.0$.**

the vectors $\boldsymbol{V}_i$ are of similar order of magnitude, the coefficients $z_i$ are gradually decreased. Thus, the final vectors of displacement terms $\boldsymbol{V}_i z_i$ are also gradually decreased. Figure 7 shows how the norm of the terms $\boldsymbol{V}_i z_i$ decreases and the series converges as the number of basis vectors is increased.

In summary, accurate results are obtained even in cases where the series of basis vectors (the binomial series) diverges. Finally, Table 4 shows that the basis vectors determined by the CA method are close to being linearly dependent even for very large changes in the design. Identical $\beta$ values are obtained for designs A, B, and C, having identical directions of change.

### B. Exact Solutions: Change in Small Number of Elements

To illustrate numerical results for cases where the number of changed elements is small, consider again the initial 10-bar truss shown in Fig. 3. The following cases of changes in the topology by deletion of members and joints have been solved (see the modified topologies in Fig. 8):

a) Deletion of members $2+6+10$ and joint 2 ($\theta =$ 33 deg).
b) Deletion of members $2+5+6+10$ and joint 2 ($\theta =$ 39 deg).
c) Deletion of members $2+6+7+10$ and joint 2 ($\theta =$ 39 deg).
d) Deletion of members $2+6+8+10$ and joint 2 ($\theta =$ 39 deg).

The exact solutions, summarized in Table 5, are found for all of these cases with only three basis vectors.

### C. Fifty-Bar Truss: Nearly Scaled Designs

Consider the cantilever truss shown in Fig. 9a. The truss is subjected to a single load at the end, and all cross-sectional areas equal unity. The modulus of elasticity is 10,000, and the 40 unknowns are the horizontal and the vertical displacements at joints 2–21, respectively. Two geometric variables have been considered, the depth $D$ and the panel width $W$. Exact solution is achieved with a single basis vector for all designs where the ratio between the depth and the width of the truss does not change. The reason is that the vertical and the horizontal joint coordinates are changed simultaneously such that the geometry is scaled. (The lengths of all members are changed by the same percentage, whereas the direction cosines are unchanged.)

With the initial geometry $D = W = 1.0$, two cases of geometrical changes are solved:

Case a. A change of 20% in the depth, $D = 1.2$ (Fig. 9b).

Case b. A change of 100% in the depth and 90% in the width, $D = 2.0$, $W = 1.9$ (Fig. 9c).

The stiffness coefficients of many members are changed; therefore exact reanalysis is not efficient. The results are given in Table 6 for first-order approximations (only two basis vectors). Comparing the results obtained for the two cases of geometrical modifications,

**Table 5 Exact displacements: various cases of deletion of members (10-bar truss)**

| Deleted members | Displacement 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $2+6$ | 2.40 | 5.80 | ——[a] | —— | −3.60 | 15.18 | −2.40 | 5.80 |
| $4+9$ | 2.11 | 4.67 | 3.30 | 13.62 | —— | 14.81 | −1.35 | 5.57 |
| $5+8+9$ | —— | —— | 2.40 | 19.76 | −3.60 | 20.96 | −3.60 | 10.38 |
| $4+5+8+9$ | 1.20 | —— | 2.40 | 19.76 | —— | 20.96 | −3.60 | 10.38 |

[a]—, Irrelevant result.

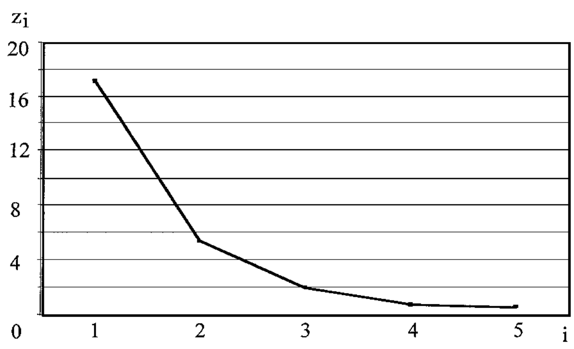


**Fig. 6 Values of $z_i$ coefficients for $\alpha = 1.0$.**

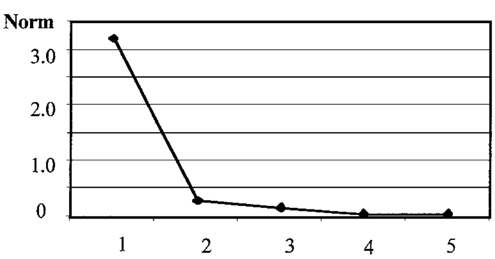


**Fig. 7 Norm of the CA terms $V_i z_i$ for $\alpha = 1.0$.**

**Table 6 First-order approximations: geometrical changes (50-bar truss)**

| Joint | Direction | Case a Exact | Case a CA1 | Case b Exact | Case b CA1 |
|---|---|---|---|---|---|
| 2 | $X$ | 0.08 | 0.09 | 0.20 | 0.20 |
| | $Y$ | 0.08 | 0.11 | 0.24 | 0.25 |
| 3 | $X$ | 0.15 | 0.16 | 0.38 | 0.38 |
| | $Y$ | 0.28 | 0.35 | 0.88 | 0.90 |
| 4 | $X$ | 0.21 | 0.22 | 0.54 | 0.54 |
| | $Y$ | 0.60 | 0.69 | 1.87 | 1.90 |
| 5 | $X$ | 0.27 | 0.26 | 0.67 | 0.68 |
| | $Y$ | 1.01 | 1.12 | 3.18 | 3.21 |
| 6 | $X$ | 0.31 | 0.29 | 0.79 | 0.79 |
| | $Y$ | 1.51 | 1.60 | 4.74 | 4.79 |
| 7 | $X$ | 0.35 | 0.32 | 0.88 | 0.88 |
| | $Y$ | 2.07 | 2.13 | 6.54 | 6.58 |
| 8 | $X$ | 0.38 | 0.34 | 0.96 | 0.95 |
| | $Y$ | 2.69 | 2.69 | 8.50 | 8.54 |
| 9 | $X$ | 0.40 | 0.35 | 1.01 | 1.01 |
| | $Y$ | 3.36 | 3.27 | 10.60 | 10.63 |
| 10 | $X$ | 0.41 | 0.35 | 1.04 | 1.04 |
| | $Y$ | 4.05 | 3.86 | 12.79 | 12.81 |
| 11 | $X$ | 0.42 | 0.35 | 1.05 | 1.05 |
| | $Y$ | 4.75 | 4.45 | 15.02 | 15.02 |

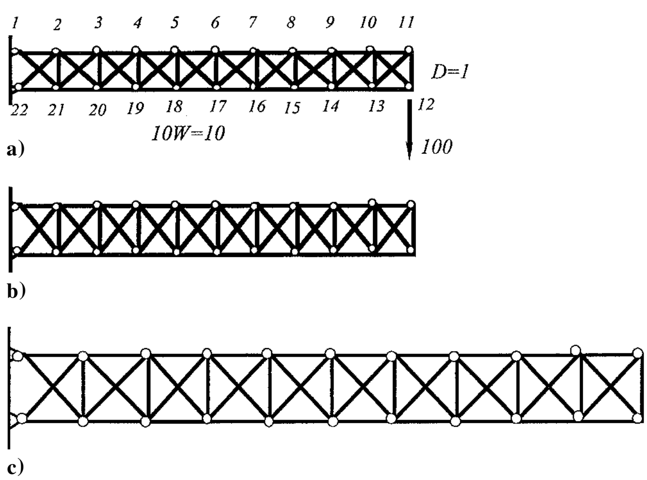


**Fig. 9 Fifty-bar truss: initial and modified geometries.**

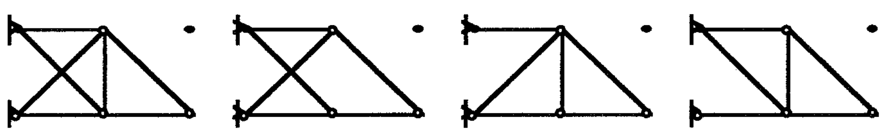

**Fig. 8 Ten-bar truss: modified topologies.**

**Table 7 Exact solution: topological changes (50-bar truss)**

| No. | Displacement |
|---|---|
| 1 | 0.09 |
| 2 | 0.14 |
| 3 | 0.17 |
| 4 | 0.46 |
| 5 | 0.24 |
| 6 | 0.93 |
| 7 | 0.30 |
| 8 | 1.55 |
| 9 | 0.35 |
| 10 | 2.29 |
| 11 | 0.39 |
| 12 | 3.13 |
| 13 | 0.42 |
| 14 | 4.05 |
| 15 | 0.44 |
| 16 | 5.03 |
| 17 | 0.45 |
| 18 | 6.04 |
| 19 | 0.45 |
| 20 | 7.07 |
| 21 | −0.55 |
| 22 | 7.07 |
| 23 | −0.54 |
| 24 | 6.03 |
| 25 | −0.52 |
| 26 | 5.02 |
| 27 | −0.49 |
| 28 | 4.04 |
| 29 | −0.45 |
| 30 | 3.12 |
| 31 | −0.40 |
| 32 | 2.28 |
| 33 | −0.34 |
| 34 | 1.54 |
| 35 | −0.27 |
| 36 | 0.92 |
| 37 | −0.19 |
| 38 | 0.45 |
| 39 | −0.10 |
| 40 | 0.13 |

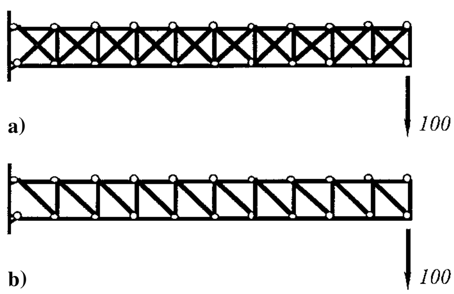


**Fig. 10 Fifty-bar truss: initial and modified topology.**

we see that better approximations are achieved in case b, involving larger changes in the geometry. This is attributed to the fact that the modified geometry is relatively close to a scaled geometry ($D = W$) for case b.

### D. Nearly Dependent Basis Vectors

Consider again the 50-bar truss shown in Fig. 10a. With 10 diagonal members deleted, the modified design is shown in Fig. 10b. Despite the relatively large number of deleted members, the exact solution shown in Table 7 is achieved with only three basis vectors. The $\cos\beta$ values obtained for the basis vectors [Eq. (30)], $\cos\beta_{1,2} = 0.9621$, $\cos\beta_{2,3} = 1.000$, show that the second and the third basis vectors determined by the CA method are linearly dependent.

## VI. Conclusions

Some typical cases where exact or accurate solutions can be achieved by the CA method have been presented and discussed. It has been shown that exact solutions are obtained when a basis vector is a linear combination of the previous vectors and also in cases of low-rank modifications to structures or scaling of the initial stiffness matrix. In general the CA method provides approximate solutions, but the results presented in the paper explain the high accuracy achieved with a small number of basis vectors. It is observed that, in many cases, where the basis vectors come close to being linearly dependent, accurate solutions are achieved. Accurate solutions are achieved also in cases of changes in a small number of elements or when the angle between the two vectors representing the initial design and modified design is small.

Several criteria for evaluating the errors involved in the approximations have been presented. In cases where the accuracy, as defined by these criteria, is insufficient it is possible to improve the results by considering additional basis vectors. The angle constraints, which are most suitable for the CA method, define a large conical region rather than a relatively small region defined by the common move limit constraints.

The main observations that have been made from the numerical examples are as follows:

1) The reduced basis coefficients can change significantly the convergence properties of the series of basis vectors (the binomial series). Accurate results are obtained by the CA method even in cases where the series of basis vectors diverges.

2) The direction of change and the magnitude of changes in the design have a significant effect on the accuracy of the approximations. Both quantities determine the value of $\theta$ that can be used as a single parameter for evaluating the effect of changes in the design.

3) For any given direction vector $\Delta X^*$, the magnitude of $\alpha$ determines the value of $\theta$. For $\theta = 0$, an exact solution is achieved by scaling the initial displacements. For a given number of basis vectors and direction of change $\Delta X^*$, the accuracy of the results depends on $\alpha$. The larger $\alpha$ is, the larger is the angle $\theta$, and more basis vectors will be needed to achieve adequate accuracy.

4) A specific $\theta$ may correspond to many different combinations of $\Delta X^*$ and $\alpha$. Various modified designs obtained by scaling a specific design correspond to the same $\theta$ angle. Similar accuracy has been achieved for these designs, even in cases where the design changes are very large.

## Acknowledgments

This work has been partially supported by the U.S. Army Automotive Research Center under the sponsorship of TACOM and by the General Motors Satellite Research Laboratory at the University of Michigan, Ann Arbor. This support is gratefully acknowledged.

## References

[1] Kirsch, U., "Efficient Reanalysis for Topological Optimization," *Structural Optimization*, Vol. 6, No. 3, 1993, pp. 143–150.

[2] Kirsch, U., "Improved Stiffness-Based First-Order Approximations for Structural Optimization," *AIAA Journal*, Vol. 33, No. 1, 1995, pp. 143–150.

[3] Kirsch, U., and Liu, S., "Exact Structural Reanalysis by a First-Order Reduced Basis Approach," *Structural Optimization*, Vol. 10, No. 3/4, 1995, pp. 153–158.

[4] Kirsch, U., and Liu, S., "Structural Reanalysis for General Layout Modifications," *AIAA Journal*, Vol. 35, No. 2, 1997, pp. 382–388.

[5] Kirsch, U., "Efficient-Accurate Approximations for Structural Optimization," *AIAA Journal*, Vol. 37, No. 12, 1999, pp. 1663–1669.

[6] Aktas, A., and Moses, F., "Reduced Basis Eigenvalue Solutions for Damaged Structures," *Mechanics of Structures and Machines*, Vol. 26, 1998, pp. 63–79.

[7] Leu, L.-J., and Huang, C.-W., "A Reduced Basis Method for Geometric Nonlinear Analysis of Structures," *IASS Journal*, Vol. 39, No. 127, 1998, pp. 71–75.

[8] Kirsch, U., "Combined Approximations—A General Approach for Structural Optimization," *Structural Optimization*, Vol. 20, No. 2, 2000, pp. 97–106.

[9] Kirsch, U., and Papalambros, P. Y., "Accurate Displacement Derivatives for Structural Optimization Using Approximate Reanalysis," *Computer Methods in Applied Mechanics and Engineering*, Vol. 190, 2001, pp. 3945–3956.

[10] Kirsch, U., and Papalambros, P. Y., "Structural Reanalysis for Topological Modifications," *Structural Optimization*, Vol. 21, No. 5, 2001, pp. 333–344.

[11]Kirsch, U., and Papalambros, P. Y., "Exact and Accurate Reanalysis of Structures for Geometrical Changes," *Engineering with Computers* (to be published).

[12]Sherman, J., and Morrison, W. J., "Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix," *Ann. Math. Statist.*, Vol. 20, 1949, p. 621.

[13]Woodbury, M., "Inverting Modified Matrices," Statistical Research Group, Princeton Univ., MR 42, Princeton, NJ, 1950.

[14]Akgun, M. A., Garcelon, J. H., and Haftka, R. T., "Fast Exact Linear and Nonlinear Structural Reanalysis and the Sherman–Morrison–Woodbury Formulas," *International Journal for Numerical Methods in Engineering*, Vol. 50, 2001, pp. 1587–1606.

[15]Kirsch, U., "Rational Move Limits for Approximate Structural Optimization," *ASCE Journal of Structural Engineering,* Vol. 123, No. 2, 1997, pp. 210–217.

[16]Nelson, S. A., and Papalambros, P. Y., "A Modified Trust Region Algorithm for Hierarchical Nonlinear Programming," *Structural Optimization*, Vol. 16, No. 1, 1998, pp. 19–28.

A. Chattopadhyay
*Associate Editor*